

CHAINS OF INFINITE ORDER, CHAINS WITH MEMORY OF VARIABLE LENGTH, AND MAPS OF THE INTERVAL.

PIERRE COLLET AND ANTONIO GALVES

ABSTRACT. We show how to construct a topological Markov map of the interval whose invariant probability measure is the stationary law of a given stochastic chain of infinite order. In particular we characterize the maps corresponding to stochastic chains with memory of variable length. The problem treated here is the converse of the classical construction of the Gibbs formalism for Markov expanding maps of the interval.

1. INTRODUCTION.

The founding papers by Bowen (2008), Ruelle (1978) and Sinai (1972) explained how to use the Gibbs formalism for Markov expanding maps of the interval. In this formalism to each such map of the interval is associated a Gibbs measure which corresponds through the dynamical coding to an absolutely continuous invariant measure. Recalling that Gibbs measures with Hölder continuous interactions are stochastic chains of infinite order (cf. Fernández and Mailard 2004 and references therein), this means that expanding maps of the interval are naturally associated to stochastic chains. In particular, piecewise affine topological Markov maps correspond to Markov chains on a finite alphabet.

In this paper we address the converse problem, namely, given a stochastic chain of infinite order, taking values on a finite alphabet, can we construct a topological Markov map of the interval whose invariant measure is the invariant probability measure of the chain?

A particular case of this question has to do with the class of stochastic chains with memory of variable length, introduced by Rissanen (1983). Recently Cénac et al. [4] have shown how to represent two interesting examples of stochastic chains with memory of variable (unbounded) length by maps of the interval. Inspired by this paper, we discuss at a more general level some of the relations between chains of infinite order, chains with memory of variable length models and expanding Markov maps of the interval.

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This paper is organized as follows. In Section 2 we briefly present the notions of expanding maps of the interval and stochastic chains of infinite order and for the convenience of the reader we recall some classical results. In Section 3 we recall the classical construction of a stochastic chain of infinite order given an expanding map of the interval. For more details about this construction we refer the reader to the articles of [16], [17] and [9] and references therein. In Section 4 we explain how to construct an expanding map of the interval given a stochastic chain of infinite order. Finally in Section 5 we study the particular case of stochastic chains with memory of variable (unbounded) length.

2. NOTATION, CHAINS AND MAPS.

In order to make this paper self contained as much as possible, we gather in this section some basic definitions and results about stochastic chains and maps of the interval.

Let A denote a finite alphabet $A = \{1, \dots, \mathcal{K}\}$. Given two integers $m \leq n$ we denote by w_m^n the sequence (w_m, \dots, w_n) of symbols in A , and A_m^n denotes the set of such sequences. Any sequence w_m^n with $m > n$ represents the empty string. The same notation is extended to the cases $m = \pm \infty$.

Given two finite sequences w and v we will denote by vw the sequence obtained by concatenating the two strings. For example, $z_{-\infty}^{-1}a$ denotes the sequence having the symbol a at the zero position and the symbols z_i at the positions $i \leq -1$.

For a finite string $a_m^n \in A_m^n$, we denote by $C(a_m^n)$ the cylinder given by

$$C(a_m^n) = \{x_{-\infty}^{+\infty} \in A_{-\infty}^{+\infty} : x_m^n = a_m^n\}.$$

2.1. Stochastic chains of infinite order. A family p of numbers $p(a|x_{-\infty}^{-1}) \in [0, 1]$, with $a \in A$ and $x_{-\infty}^{-1} \in A_{-\infty}^{-1}$, is called a family of transition probabilities if it satisfies the two conditions

- For each fixed sequence $x_{-\infty}^{-1}$

$$\sum_a p(a|x_{-\infty}^{-1}) = 1.$$

- For each symbol $a \in A$, the map

$$x_{-\infty}^{-1} \longrightarrow p(a|x_{-\infty}^{-1})$$

is measurable with the product sigma-algebra on $A_{-\infty}^{-1}$.

Definition 2.1. A family p of transition probabilities satisfies the condition of non-nullness if

$$\inf\{p(a|x_{-\infty}^{-1}) : a \in A, x_{-\infty}^{-1} \in A_{-\infty}^{-1}\} > 0.$$

Definition 2.2. *The continuity rate of a family p of transition probabilities is the sequence $(\beta_k)_{k \geq 1}$ defined by*

$$\beta_k = \sup \left\{ |p(a|x_{-\infty}^{-1}) - p(a|y_{-\infty}^{-1})| : a \in A, x_{-\infty}^{-1}, y_{-\infty}^{-1} \in A_{-\infty}^{-1} \text{ with } x_{-k}^{-1} = y_{-k}^{-1} x_{-\infty}^{-1} \stackrel{k}{=} y_{-\infty}^{-1} \right\}.$$

Definition 2.3. *The family p of transition probabilities with continuity rate (β_k) is said to be continuous if*

$$\lim_{k \rightarrow +\infty} \beta_k = 0.$$

Definition 2.4. *We will say that a probability measure \mathbb{P} on $A^{\mathbb{Z}}$ is translation invariant (or stationary) if for any $m \geq 0$ and for any $a_0^m \in A_0^m$, we have*

$$\mathbb{P}\{X_n^{n+m} = a_0^m\} = \mathbb{P}\{X_0^m = a_0^m\}$$

for any $n \in \mathbb{Z}$.

The notion of translation invariance says that the probability measure \mathbb{P} is invariant with respect to the shift $\mathcal{S} : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$, defined as follows. For every sequence $\underline{z} = z_{-\infty}^{+\infty}$, we have

$$\mathcal{S}(\underline{z})_i = \underline{z}_{i-1}.$$

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Definition 2.5. *We will say that a probability measure \mathbb{P} on $A^{\mathbb{Z}}$ is invariant with respect to p , if for any continuous function $f : A_{-\infty}^0 \rightarrow \mathbb{R}$ we have*

$$(2.1) \quad \int f(z_{-\infty}^0) d\mathbb{P}(z_{-\infty}^0) = \int \sum_{a \in A} p(a | (z_{-\infty}^{-1}) f(z_{-\infty}^{-1} a) d\mathbb{P}(z_{-\infty}^{-1})$$

From stationarity and invariance of \mathbb{P} with respect to p it follows immediately that for any pair $m \leq n$ of integers and for any $a_m^n \in A_m^n$, we have

$$\mathbb{P}(C(a_m^n)) = \int_{A_{-\infty}^{m-1}} \prod_{j=m}^n p(a_j | a_m^{j-1} x_{-\infty}^{m-1}) d\mathbb{P}(x_{-\infty}^{m-1}).$$

For later references, it is convenient to collect in the following theorem some well know results about families of transition probabilities.

Theorem 2.6. *If the family of transition probabilities satisfies the non-nullness condition 2.1 and the sequence of continuity rates is summable, then there exists a unique ergodic stationary probability measure \mathbb{P} on $A^{\mathbb{Z}}$, invariant with respect to p . This invariant probability measure has no atom, and for any finite sequence a_m^n , $\mathbb{P}(C(a_m^n)) > 0$.*

This type of result has been proved by many authors starting with Onicescu and Mihoc (1935), Doeblin and Fortet (1937) and Harris (1955) and Comets et al. (2002).

Let us now consider the probability space having $A^{\mathbb{Z}}$ as sample space, equipped with its product σ -algebra, and having \mathbb{P} , whose existence is granted by Theorem 2.6, as probability measure. We can define a stochastic chain $(X_n)_{n \in \mathbb{Z}}$ on this probability space, by taking, for each $n \in \mathbb{Z}$

$$X_n : A^{\mathbb{Z}} \longrightarrow A$$

as the projection on the n^{th} coordinate. In other words, for any $m \leq n$ and any choice of the sequence a_m^n , we have

$$\mathbb{P}(C(a_m^n)) = \mathbb{P}\{X_m^n = a_m^n\}.$$

The stochastic chain $(X_n)_{n \in \mathbb{Z}}$ is said to be associated to the family of transition probabilities p .

2.2. Piecewise expanding maps of the interval. From now on let $\Omega = [0, 1]$. We first recall the definition of a piecewise expanding map of the interval Ω . Let $0 = \eta_0 < \eta_1 < \dots < \eta_{\mathcal{K}} = 1$ be a finite sequence and for each interval $I_j =]\eta_{j-1}, \eta_j[$, with $(1 \leq j \leq \mathcal{K})$, let T_j be a monotone map from I_j to Ω which extends to a C^2 map on $\bar{I}_j = [\eta_{j-1}, \eta_j]$. The map T is defined as follows. For each $\omega \in \Omega \setminus \{\eta_0, \eta_1, \dots, \eta_{\mathcal{K}}\}$

$$T(\omega) = T_j(\omega), \quad \text{if } \omega \in I_j.$$

We denote by \mathcal{P} , the collection of open intervals I_j , with $j = 1, \dots, \mathcal{K}$, and observe that it defines a partition of the $\Omega \setminus \mathcal{N}_0$, where $\mathcal{N}_0 = \{\eta_0, \eta_1, \dots, \eta_{\mathcal{K}}\}$. From now on let us call $A = \{1, \dots, \mathcal{K}\}$ the set of indexes of the partition \mathcal{P} .

Definition 2.7. *The map T of the interval has the (uniform) expanding property if there is an integer $m > 0$ and a constant $c > 1$ such that at any point where T^m is differentiable we have*

$$|T^{m'}| \geq c.$$

Definition 2.8. *The piecewise expanding map T of the interval is said to be topological Markov if for any $i = 1, \dots, \mathcal{K}$, the closure of $T(I_i)$ is a union of closures of intervals I_j , $j \in \{1, \dots, \mathcal{K}\}$. The map T is called full topological Markov if for any $i = 1, \dots, \mathcal{K}$.*

Note that the topological Markov property notion is not to be mistaken with the Markov property of stochastic processes.

Recall that the map T is not defined on the finite set \mathcal{N}_0 . Call \mathcal{N} the set of pre-images of \mathcal{N}_0 , namely

$$\mathcal{N} = \mathcal{N}_0 \cup \bigcup_{k \geq 1} \{\omega \in \Omega \mid T^k(\omega) \in \mathcal{N}_0\}$$

Given an full topological Markov expanding map of the interval T , we define a coding of $\Omega \setminus \mathcal{N}$ with alphabet $A = \{1, \dots, \mathcal{K}\}$. This coding is a map W from $\Omega \setminus \mathcal{N}$ to $A^{\mathbb{N}}$

$$\omega \longrightarrow W(\omega) = (W_n(\omega))_{n \in \mathbb{N}}$$

given by

$$W_n(\omega) = j, \quad \text{if} \quad T^n(\omega) \in I_j.$$

Given a full topological Markov expanding map of the interval T , we have just associated a code to a point in $\Omega \setminus \mathcal{N}$. We can also go in the opposite direction and this is the content of the next proposition. To simplify the presentation we will restrict ourselves to the case of full Markov maps. The extension to the case of general Markov maps is straightforward.

Proposition 2.9. *Assume T is a full topological Markov expanding map of the interval and A is the set of indexes of the partition \mathcal{P} . Then given a code $x_0^{+\infty} \in A_0^{+\infty}$, there exists at most one point in the interval Ω which is coded by this sequence.*

Both directions are well known, see for instance [16] and [17].

3. CONSTRUCTING A CHAIN FROM A MAP

Let μ be a T -invariant measure defined on $\Omega = [0, 1]$. We now have the three ingredients of a probability space, namely the sample space $\Omega = [0, 1]$, with its Borel σ -algebra and the T -invariant probability μ . Furthermore, the coding associated to the map T defines a sequence of random variables $(W_n)_{n \in \mathbb{N}}$ with values in the alphabet A .

Let us denote by $q(x_m^n)$ (with $m \leq n$ belong to \mathbb{Z}) the cylinder probabilities on $A^{\mathbb{Z}}$ defined by

$$q(x_m^n) = \mu\{\omega \in \Omega, W_0(\omega) = x_n, W_1(\omega) = x_{n-1}, \dots, W_{n-m}(\omega) = x_m\}.$$

The time was reversed in the definition of q to follow the usual convention for stochastic processes.

Kolmogorov's Existence Theorem implies that there exists a unique stationary probability measure \mathbb{P} on $A^{\mathbb{Z}}$ such that for any integers $m \leq n$, and any sequence $x_m^n \in A_m^n$ we have

$$\mathbb{P}\{X_m^n = x_m^n\} = q(x_m^n),$$

where $X_n : A^{\mathbb{Z}} \rightarrow A$ is the projection on the n^{th} coordinate. The $(X_n)_{n \in \mathbb{Z}}$ is in general a chain of infinite order. The next theorem will give an explicit expression for its family of transition probabilities

$$p(b \mid a_{-\infty}^{-1}) = \mathbb{P}(X_0 = b \mid X_{-\infty}^{-1} = a_{-\infty}^{-1}).$$

We denote by λ the Lebesgue measure of the interval $\Omega = [0, 1]$.

Theorem 3.1. *Let $\Omega = [0, 1]$ and let T be a full topological Markov expanding map of the interval. Assume that the Lebesgue measure λ is invariant and ergodic with respect to T . Then the family of transition probabilities of the associated chain of infinite order is given by*

$$p(b \mid a_{-\infty}^{-1}) = \lim_{n \rightarrow \infty} \frac{1}{|T'(\omega_n)|}$$

where ω_n is any point in Ω , such that

$$W_0^n(\omega_n) = (b, a_{-1}, \dots, a_{-n}) .$$

For the proof of Theorem 3.1 we refer the reader to [16], [17] and [10].

In the general case, where the invariant absolutely continuous invariant measure μ is not the Lebesgue measure λ , we have the following result.

Corollary 3.2. *Let T be a topological Markov piecewise expanding map on Ω . Assume that the probability measure μ which is absolutely continuous with respect to the Lebesgue measure λ is invariant and ergodic with respect to T . Then the family of transition probabilities of the associated chain of infinite order is given by*

$$p(b \mid a_{-\infty}^{-1}) = \lim_{n \rightarrow \infty} \frac{g(\omega_n)}{g(T(\omega_n))|T'(\omega_n)|}$$

where $g = d\mu/d\lambda$, and ω_n is any point such that

$$W_0^n(\omega_n) = (b, a_{-1}, \dots, a_{-n}) .$$

Proof. Let G be the distribution of μ defined in the usual way by

$$G(t) = \mu([0, t]) .$$

Obviously G is a non decreasing function which is also continuous since μ is absolutely continuous with respect to the Lebesgue measure λ . By a theorem of Buzzi (1997) (see also Liverani (1995)), μ is equivalent to the λ , and $g = d\mu/d\lambda = G'$ is a continuous non-vanishing function. In other words G is a C^1 diffeomorphism.

Consider G^{-1} as a random variable defined on the probability space $(\Omega, \mathcal{P}_\infty, \lambda)$. This fact together with the invertibility of G implies that the Lebesgue measure λ is invariant and ergodic with respect to the map T_0 defined by

$$T_0 = G \circ T \circ G^{-1} .$$

Theorem 3.1 applies to T_0 , and the corollary follows by the chain rule. \square

4. CONSTRUCTING A MAP FROM A CHAIN.

Let (X_n) be a stationary ergodic stochastic chain taking values in the finite alphabet $A = \{1, \dots, \mathcal{K}\}$, and defined on a probability space $(A^{\mathbb{Z}}, \mathcal{F}, \mathbb{P})$. Let us assume that the law \mathbb{P} of the chain has no atom. Our goal is to define a map $T : \Omega \rightarrow \Omega$, where $\Omega = [0, 1]$, such that the construction of Section 3 recovers the chain (X_n) . The map T will be defined by a conjugation to the shift \mathcal{S} through a map $h : A_{-\infty}^0 \rightarrow \Omega$ defined below.

We define a distance on $A_{-\infty}^0$ as follows.

Definition 4.1. *First of all, for two sequences $x_{-\infty}^0$ and $y_{-\infty}^0$, denote by $\delta(x_{-\infty}^0, y_{-\infty}^0)$ the nearest position to the origin where these two sequences differ, namely*

$$\delta(x_{-\infty}^0, y_{-\infty}^0) = \min \{n \geq 0 : x_{-n} \neq y_{-n}\} .$$

For a fixed number $0 < \zeta < 1$, we define the distance d on $A_{-\infty}^0$ by

$$d(x_{-\infty}^0, y_{-\infty}^0) = \zeta^{\delta(x_{-\infty}^0, y_{-\infty}^0)} .$$

We denote by $<$ the lexicographic order on $A_{-\infty}^0$. Namely $x_{-\infty}^0 < y_{-\infty}^0$, if for some $m \geq 0$, we have $x_{-(m-1)}^0 = y_{-(m-1)}^0$ and $x_{-m}^0 < y_{-m}^0$. For a point $x_{-\infty}^0 \in A_{-\infty}^0$, we denote by $J(x_{-\infty}^0)$ the set of points

$$J(x_{-\infty}^0) = \{y_{-\infty}^0 \mid y_{-\infty}^0 \leq x_{-\infty}^0\} .$$

We define the map h from $A_{-\infty}^0$ to Ω by

$$h(x_{-\infty}^0) = \mathbb{P}(J(x_{-\infty}^0)) .$$

Before stating the properties of the map h , we need to define a countable set \mathcal{Q} of exceptional codes, given by

$$\mathcal{Q} = \bigcup_{j=1}^{\mathcal{K}-1} \{\mathcal{K}_{-\infty}^{-1}j, 1_{-\infty}^{-1}(j+1)\} \bigcup_{k=0}^{\infty} \bigcup_{x_{-k}^0 \in A_{-k}^0} \bigcup_{j=1}^{\mathcal{K}-1} \{\mathcal{K}_{-\infty}^{-1}jx_{-k}^0, 1_{-\infty}^{-1}(j+1)x_{-k}^0\} ,$$

where $\mathcal{K}_{-\infty}^{-1}$ and $1_{-\infty}^{-1}$ denote the sequences identically equal to \mathcal{K} and 1, respectively.

Proposition 4.2. *Let p be a family of transition probabilities satisfying the non-nullness condition 2.1. The map h defined above has the following properties*

- i) h is non decreasing on Ω and strictly increasing outside \mathcal{Q} ;
- ii) h is continuous;
- iii) h is invertible except on the countable set $h(\mathcal{Q})$, and the set of preimages of any point in $h(\mathcal{Q})$ has cardinality at most two;
- iv) the inverse function h^{-1} is continuous outside $h(\mathcal{Q})$;
- v) the image of \mathbb{P} by h is the Lebesgue measure on Ω ;
- vi) finally h is surjective.

Proof. We first prove that the map h is injective except on the countable set \mathcal{Q} . Let $x_{-\infty}^0 < y_{-\infty}^0$. This means that $x_0 < y_0$, or there exists an integer $k \geq 0$ such that $x_{-k}^0 = y_{-k}^0$, and $x_{-(k+1)} < y_{-(k+1)}$. Assume $x_{-\infty}^0 \notin \mathcal{Q}$. This implies that for infinitely many indices n , we have $x_{-n} \leq \mathcal{K} - 1$. Let $m > k$ be such an index. For any $z_{-\infty}^0$ in the cylinder $C(\mathcal{K}x_{-(m-1)}^0)$ we have

$$x_{-m}^0 < z_{-m}^0 < y_{-m}^0 .$$

Therefore

$$J(x_{-m}^0) \cap C(\mathcal{K}x_{-(m-1)}^0) = \emptyset , \quad \text{and} \quad C(\mathcal{K}x_{-(m-1)}^0) \subset J(y_{-m}^0) .$$

From Theorem 2.6 we have $\mathbb{P}(C(\mathcal{K}x_{-(m-1)}^0)) > 0$, hence

$$h(x_{-m}^0) = \mathbb{P}(J(x_{-m}^0)) < \mathbb{P}(J(y_{-m}^0)) = h(y_{-m}^0) .$$

The case where $y_{-\infty}^0 \notin \mathcal{Q}$ can be treated similarly.

If $x_{-\infty}^0 \in \mathcal{Q}$ and $y_{-\infty}^0 \in \mathcal{Q}$ but

$$(x_{-\infty}^0, y_{-\infty}^0) \neq (\mathcal{K}_{-\infty}^{-1}x_0, 1_{-\infty}^{-1}(x_0 + 1))$$

and for any $k \geq 0$,

$$y_{-\infty}^0 \neq 1_{-\infty}^{-k-1}(x_{-k-1} + 1)x_{-k}^0$$

then there exists $\tilde{x}_{-\infty}^0 \notin \mathcal{Q}$ and such that $x_{-\infty}^0 < \tilde{x}_{-\infty}^0 < y_{-\infty}^0$. From above it follows that

$$h(x_{-\infty}^0) < h(\tilde{x}_{-\infty}^0) < h(y_{-\infty}^0) .$$

Finally, if for some $a \in \{1, \dots, \mathcal{K} - 1\}$ we have either

$$x_{-\infty}^0 = \mathcal{K}_{-\infty}^{-1}a \quad \text{and} \quad y_{-\infty}^0 = 1_{-\infty}^{-1}(a + 1) ,$$

or

$$x_{-\infty}^0 = \mathcal{K}_{-\infty}^{-(k+2)}ax_{-k}^0 \quad \text{and} \quad y_{-\infty}^0 = 1_{-\infty}^{-(k+2)}(a + 1)x_{-k}^0 ,$$

for some $k \geq 1$, then $h(x_{-\infty}^0) = h(y_{-\infty}^0)$. This concludes the proof of (i).

Now let us prove that the map h is continuous. Take $x_{-\infty}^0 \in A_{-\infty}^0$, and let $(y_{-\infty}^0(n))$ be a sequence in $A_{-\infty}^0$ converging to $x_{-\infty}^0$ in the metric defined in 4.1. This implies that for any k there exists $\bar{n}(k)$ such that for any $n \geq \bar{n}(k)$, $y_{-k}^0(n) = x_{-k}^0$. This implies

$$J(y_{-\infty}^0(n)) \Delta J(x_{-\infty}^0) \subset C(x_{-k}^0) ,$$

and therefore

$$|h(y_{-\infty}^0(n)) - h(x_{-\infty}^0)| \leq \mathbb{P}(C(x_{-k}^0)) .$$

By Theorem 2.6 the probability measure \mathbb{P} has no atoms, hence $\mathbb{P}(C(x_{-k}^0))$ tends to 0, when k tends to ∞ , proving that h is continuous. This concludes the proof of (ii).

Assertion (iii) and (iv) follow immediately from (i) and (ii).

Finally to prove (v), take $z \in \Omega \setminus h(\mathcal{Q})$. The inverse value $h^{-1}(z)$ is uniquely defined, and therefore

$$\lambda([0, z]) = z = h(h^{-1}(z)) = \mathbb{P}(J(h^{-1}(z))) = \mathbb{P}(h^{-1}([0, z])) .$$

Since the measure \mathbb{P} and the Lebesgue measure have no atoms, the same result holds for the countable set of points in $h(\mathcal{Q})$. This implies by standard measure theoretic arguments (see for example Breiman 1992) that λ is the image of \mathbb{P} by h . This concludes the proof of (v).

Finally (vi) follows from the fact that the measure \mathbb{P} has no atom by Theorem 2.6 the map h is continuous and hence surjective. \square

We define the map T on $\Omega \setminus \mathcal{Q}$ by

$$T = h \circ \mathcal{S} \circ h^{-1} .$$

More explicitly, for $z \in \Omega \setminus \mathcal{Q}$ we have

$$(4.1) \quad T(z) = \mathbb{P}(J(\mathcal{S}h^{-1}(z))) = \mathbb{P}(\mathcal{S}J(h^{-1}(z))) .$$

Theorem 4.3. *Let p be a family of transition probabilities satisfying the non-nullness and the continuity conditions 2.1 and 2.3. Then*

- (1) *The map T defined above can be continuously extended to a monotone increasing map on each interval $I_j =]\eta_{j-1}, \eta_j[$, with $j = 1, \dots, \mathcal{K}$, with end points $0 = \eta_0 < \eta_1 < \dots < \eta_{\mathcal{K}} = 1$ defined by*

$$\eta_k = h(\mathcal{K}_{-\infty}^{-1}k) = h(1_{-\infty}^{-1}(k+1)) , \text{ for } k = 1, \dots, \mathcal{K} - 1 .$$

- (2) *The extended map (also denoted by T) is a topological Markov map and the Lebesgue measure is invariant by T and ergodic. Moreover the regular versions of the conditional probabilities associated to the sequence of dynamical partitions are given by p .*

- (3) *The map T is differentiable outside $h(\mathcal{Q})$ and for each $\omega \in \Omega \setminus h(\mathcal{Q})$ we have*

$$T'(\omega) = \frac{1}{p(h^{-1}(\omega)_0 \mid h^{-1}(\omega)_{-\infty}^{-1})} .$$

In this formula we denote the successive elements of the sequence $h^{-1}(\omega) \in A_{-\infty}^0$ by $h^{-1}(\omega)_{-\infty}^0$.

- (4) *For $\omega \in h(\mathcal{Q})$, with*

$$\omega = h(\mathcal{K}_{-\infty}^{-(k+2)} a z_{-k}^0) = h(1_{-\infty}^{-(k+2)} (a+1) z_{-k}^0) ,$$

for some $a \in \{1, \dots, \mathcal{K} - 1\}$ and some integer $k \geq -1$, then the left and right derivatives of T at ω exist and are given by

$$\frac{1}{p(z_0 \mid z_{-k}^{-1} a \mathcal{K}_{-\infty}^{-(k+2)})} \quad \text{and} \quad \frac{1}{p(z_0 \mid z_{-k}^{-1} (a+1) 1_{-\infty}^{-(k+2)})} ,$$

respectively.

(5) In particular, if p is such that for any $a \in \{1, \dots, \mathcal{K} - 1\}$ and any integer $k \geq 0$ and any z_{-k}^0 , we have

$$(4.2) \quad p(z_0 \mid z_{-k}^{-1} a \mathcal{K}_{-\infty}^{-(k+2)}) = p(z_0 \mid z_{-k}^{-1} (a+1) 1_{-\infty}^{-(k+2)}),$$

then the map T is piecewise C^1 .

(6) If the continuity rate β_k , defined in 2.2, decays exponentially fast, and conditions (4.2) are satisfied, then the map T is piecewise $C^{1+\alpha}$, where $\alpha > 0$ depends on the exponential rate of decay of β_k .

The proof of Theorem 4.3 will use several times the following lemma

Lemma 4.4. *For any pair of points $u < v$ in $\Omega \setminus h(\mathcal{Q})$ and belonging to the same monotonicity interval $I_j =]\eta_{j-1}, \eta_j[$ of T , for any $j = 1, \dots, \mathcal{K} - 1$, we have*

$$T(v) - T(u) = \int_u^v \frac{\lambda(d\omega)}{p(h^{-1}(\omega)_0 \mid h^{-1}(\omega)_{-\infty}^{-1})}.$$

Proof. By definition

$$(4.3) \quad \begin{aligned} \int_u^v \frac{d\lambda(\omega)}{p(h^{-1}(\omega)_0 \mid h^{-1}(\omega)_{-\infty}^{-1})} &= \int_{h^{-1}([u,v])} \frac{d\mathbb{P}(z_{-\infty}^0)}{p(z_0 \mid z_{-\infty}^{-1})} \\ &= \int \frac{\mathbb{1}_{J(h^{-1}(v)) \setminus J(h^{-1}(u))}(z_{-\infty}^0)}{p(z_0 \mid z_{-\infty}^{-1})} d\mathbb{P}(z_{-\infty}^0). \end{aligned}$$

In the above formula, $\mathbb{1}_{J(h^{-1}(v)) \setminus J(h^{-1}(u))}$ denotes the characteristic function of the set $J(h^{-1}(v)) \setminus J(h^{-1}(u))$. Since u and v by hypothesis belong to the same monotonicity interval, we have that

$$h^{-1}(u)_0 = h^{-1}(v)_0.$$

Let $f : A_{-\infty}^0 \rightarrow \mathbb{R}$ be the function

$$f(z_{-\infty}^0) = \frac{\mathbb{1}_{J(h^{-1}(v)) \setminus J(h^{-1}(u))}(z_{-\infty}^0)}{p(z_0 \mid z_{-\infty}^{-1})} = \frac{\mathbb{1}_{\{z_0 = h^{-1}(v)_0\}} \mathbb{1}_{\mathcal{S}(J(h^{-1}(v)) \setminus J(h^{-1}(u)))}(z_{-\infty}^{-1})}{p(z_0 \mid z_{-\infty}^{-1})}.$$

Using the invariance of \mathbb{P} (see (2.1)) with the function f , we can rewrite the integral (4.3) as

$$\int \frac{\mathbb{1}_{J(h^{-1}(v)) \setminus J(h^{-1}(u))}(z_{-\infty}^0)}{p(z_0 \mid z_{-\infty}^{-1})} d\mathbb{P}(z_{-\infty}^0) = \int \mathbb{1}_{\mathcal{S}(J(h^{-1}(v)) \setminus J(h^{-1}(u)))}(z_{-\infty}^{-1}) d\mathbb{P}(z_{-\infty}^{-1}).$$

Now we observe that

$$\mathcal{S}(J(h^{-1}(u))) \subset \mathcal{S}(J(h^{-1}(v)))$$

and therefore

$$\int \mathbb{1}_{\mathcal{S}(J(h^{-1}(v)) \setminus J(h^{-1}(u)))}(z_{-\infty}^{-1}) d\mathbb{P}(z_{-\infty}^{-1}) = \mathbb{P}\{\mathcal{S}(J(h^{-1}(v)))\} - \mathbb{P}\{\mathcal{S}(J(h^{-1}(u)))\}.$$

Now it is enough to use equality (4.1) to conclude the proof. \square

We can now prove Theorem 4.3.

Proof. Assertion 1 of the theorem follows directly from Lemma 4.1.

For Assertion 2, we start by observing that for $i = 1, \dots, \mathcal{K}$ we have

$$\lim_{\omega \nearrow \eta_i} T(\omega) = 1$$

and for $i = 0, \dots, \mathcal{K} - 1$

$$\lim_{\omega \searrow \eta_i} T(\omega) = 0 .$$

The topological Markov property follows from the piecewise monotonicity of T .

The invariance and ergodicity of the Lebesgue measure λ follows from the fact that T and the shift \mathcal{S} are conjugated by h .

To prove that p is the regular version of the conditional probability we start with equality

$$\lambda(I_{x_{-k}^0}) = \mathbb{P}(C(x_{-k}^0))$$

where

$$I_{x_{-k}^0} = \{\omega \mid T^j(\omega) \in I_{x_{-j}}, j = 0 \dots k\} .$$

Therefore, for any $x_{-\infty}^0 \in A_{-\infty}^0$

$$\lim_{k \rightarrow \infty} \frac{\lambda(I_{x_{-k}^0})}{\lambda(I_{x_{-k}^{-1}})} = \lim_{k \rightarrow \infty} \frac{\lambda(I_{x_{-k}^0})}{\lambda(T(I_{x_{-k}^0}))} = \lim_{k \rightarrow \infty} \frac{\mathbb{P}(C(x_{-k}^0))}{\mathbb{P}(C(x_{-k}^{-1}))} = p(x_0 \mid x_{-\infty}^{-1}) ,$$

where the last equality follows from the continuity of the family of transition probabilities p .

Assertions 3, 4 and 5 follow directly from Lemma 4.4, and the finiteness of the derivative follows from the non-nullness assumption.

To prove Assertion 6, we first observe that the exponential decay of the continuity rate β_k implies that there exists two constants $C > 0$ and $0 < \rho < 1$ such that for any $k \geq 1$

$$(4.4) \quad \beta_k \leq C \rho^k .$$

Let

$$\gamma = \frac{1}{\sup_{x_{-\infty}^0 \in A_{-\infty}^0} p(x_0 \mid x_{-\infty}^{-1})} ,$$

and

$$\Gamma = \frac{1}{\inf_{x_{-\infty}^0 \in A_{-\infty}^0} p(x_0 \mid x_{-\infty}^{-1})} .$$

From the non-nullness assumption it follows immediately that $\gamma > 1$, and $\Gamma < \infty$. For ω and ω' in the same interval of monotonicity I_j , let

$$m = \delta(h^{-1}(\omega), h^{-1}(\omega')) ,$$

where δ was defined in 4.1. Let

$$M = \left\lceil -\frac{\log |\omega - \omega'|}{\log \gamma} \right\rceil ,$$

where $[\]$ denotes the integer part.

We first consider the case $m > M$. Then from (4.4) we have

$$\begin{aligned} |p(h^{-1}(\omega)_0 | h^{-1}(\omega)_{-\infty}^{-1}) - p(h^{-1}(\omega')_0 | h^{-1}(\omega')_{-\infty}^{-1})| &\leq C \rho^{\delta(h^{-1}(\omega), h^{-1}(\omega'))} \\ &\leq C \rho^{-1} \rho^{-\log |\omega - \omega'| / \log \gamma} = C \rho^{-1} |\omega - \omega'|^{-\log \rho / \log \gamma}. \end{aligned}$$

This implies that

$$\begin{aligned} |T'(\omega) - T'(\omega')| &= \left| \frac{1}{p(h^{-1}(\omega)_0 | h^{-1}(\omega)_{-\infty}^{-1})} - \frac{1}{p(h^{-1}(\omega')_0 | h^{-1}(\omega')_{-\infty}^{-1})} \right| \\ &\leq \Gamma^2 C \rho^{-1} |\omega - \omega'|^{-\log \rho / \log \gamma}. \end{aligned}$$

We now consider the case $m \leq M$. If

$$|\omega - \omega'|^{1/2} \Gamma^m > \min \{ \lambda(I_1), \lambda(I_K) \},$$

we have

$$m \geq -\frac{1}{2 \log \Gamma} \log |\omega - \omega'| + \frac{\log \min \{ \lambda(I_1), \lambda(I_K) \}}{\log \Gamma}.$$

The same estimate as before implies

$$|T'(\omega) - T'(\omega')| \leq \Gamma^2 C \rho^{-1} \rho^{\log \min \{ \lambda(I_1), \lambda(I_K) \} / \log \Gamma} |\omega - \omega'|^{-\log \rho / (2 \log \gamma)}.$$

Finally if

$$|\omega - \omega'|^{1/2} \Gamma^m \leq \min \{ \lambda(I_1), \lambda(I_K) \},$$

we have, assuming $\omega' > \omega$, that

$$h^{-1}(\omega) = h^{-1}(\omega)_{-\infty}^{-m-2-M/2} \mathcal{K}_{-m-1-M/2}^{-m-1} h^{-1}(\omega)_{-m} h^{-1}(\omega)_{-m+1}^0$$

and

$$h^{-1}(\omega') = h^{-1}(\omega')_{-\infty}^{-m-2-M/2} 1_{-m-1-M/2}^{-m-1} (h^{-1}(\omega)_{-m} + 1) h^{-1}(\omega)_{-m+1}^0.$$

From inequality (4.4) we get

$$\begin{aligned} |p(h^{-1}(\omega)_0 | h^{-1}(\omega)_{-m+1}^{-1} h^{-1}(\omega)_{-m} \mathcal{K}_{-m-1-M/2}^{-m-1} h^{-1}(\omega)_{-\infty}^{-m-2-M/2}) \\ - p(h^{-1}(\omega)_0 | h^{-1}(\omega)_{-m+1}^{-1} h^{-1}(\omega)_{-m} \mathcal{K}_{-\infty}^{-m-1})| \leq C \rho^{m+M/2} \end{aligned}$$

and

$$\begin{aligned} |p(h^{-1}(\omega')_0 | h^{-1}(\omega')_{-m+1}^{-1} h^{-1}(\omega')_{-m} 1_{-m-1-M/2}^{-m} h^{-1}(\omega')_{-\infty}^{-m-2-M/2}) \\ - p(h^{-1}(\omega')_0 | h^{-1}(\omega')_{-m+1}^{-1} h^{-1}(\omega')_{-m} 1_{-\infty}^{-m-1})| \leq C \rho^{m+M/2}. \end{aligned}$$

Observing that

$$\begin{aligned} h^{-1}(\omega)_{-m+1}^0 &= h^{-1}(\omega')_{-m+1}^0, \\ h^{-1}(\omega')_{-m} &= h^{-1}(\omega)_{-m} + 1, \end{aligned}$$

and using Assumption 4.2, we obtain

$$\begin{aligned} |p(h^{-1}(\omega)_0 | h^{-1}(\omega)_{-m+1}^{-1} h^{-1}(\omega)_{-m} \mathcal{K}_{-m-1-M/2}^{-m-1} h^{-1}(\omega)_{-\infty}^{-m-2-M/2}) \\ - p(h^{-1}(\omega')_0 | h^{-1}(\omega')_{-m+1}^{-1} h^{-1}(\omega')_{-m} 1_{-m-1-M/2}^{-m} h^{-1}(\omega')_{-\infty}^{-m-2-M/2})| \\ \leq 2 C \rho^{m+M/2}. \end{aligned}$$

The conclusion follows as in the two other cases. \square

5. THE CASE OF CHAINS WITH MEMORY OF VARIABLE LENGTH.

Stochastic chains with memory of variable length appeared in the pioneering paper by Rissanen (1983) as a universal system for data compression. We briefly recall the definition of this class of stochastic chains.

Given a finite alphabet A , we define the basic notion of *context tree*.

Definition 5.1. *A set of strings*

$$\tau \subset \bigcup_{k \geq 1} A_{-k}^{-1} \bigcup A_{-\infty}^{-1}$$

is a context tree if

- (1) $\bigcup_{w \in \tau} C(w) = A_{-\infty}^{-1}$;
- (2) for any pair w and w' of elements of τ , if $w \neq w'$, then $C(w) \cap C(w') = \emptyset$.

In the above definition w and w' denote two sequences, either finite or infinite, and $C(w)$ is the set of all elements of $A_{-\infty}^{-1}$ having the string w as a suffix, i. e. having w as final sequence. In case w is finite, $C(w)$ is a cylinder. In case w is infinite $C(w)$ is the unitary set whose unique element is w . The name *context tree* comes from the fact that τ can be described by the leaves of a rooted tree. The strings belonging to τ are called *contexts*.

Definition 5.2. *A probabilistic context tree is a pair (τ, p) , where τ is a context tree and*

$$p = \{p(\cdot | w) | w \in \tau\}$$

is a family indexed by τ of probability measures on the set A .

Given a probabilistic context tree (τ, p) , we define a family of infinite order transition probabilities \tilde{p} on A as follows. For any sequence $x_{-\infty}^{-1} \in A_{-\infty}^{-1}$, and for any symbol $a \in A$

$$(5.1) \quad \tilde{p}(a | x_{-\infty}^{-1}) = p(a | w)$$

where w is the unique element of τ , such that $x_{-\infty}^{-1} \in C(w)$.

Definition 5.3. *A stochastic chain of infinite order is said to have a memory of variable length described by a probabilistic context tree (τ, p) if its family of transition probabilities satisfies conditions (5.1).*

Intuitively speaking in a chain with memory of variable length, at each time step, to predict the next symbol, it is enough to use the past steps corresponding to the context associated to this past.

The question we address in this section is to characterize the maps associated to transition probabilities defined by a probabilistic context tree. This is the content of the following theorem.

Theorem 5.4. *Let T be a topological Markov expanding map of the interval with alphabet of monotonicity intervals A , and with the Lebesgue measure invariant and ergodic. Assume there is a tree of contexts τ on the alphabet A such that*

$$\sum_{k=1}^{\infty} \sum_{x_{-k}^{-1} \in \tau \cap A_{-k}^{-1}} \lambda(C(x_{-k}^{-1})) = 1 ,$$

and for any $x_{-k}^{-1} \in \tau$, for any $a \in A$ and for any ω and ω' satisfying

$$W(\omega)_0^k = a x_{-1} \dots x_{-k} , \quad \text{and} \quad W(\omega')_0^k = a x_{-1} \dots x_{-k} ,$$

we have

$$T'(\omega) = T'(\omega') .$$

Then the family of transition probabilities associated to the map T by theorem 3.1 is a chain with variable length whose contexts are almost surely finite.

Conversely, given a family of transition probabilities which is a chain of variable length with almost surely finite contexts (for an invariant measure), then the associated map by (4.1) (see also Theorem 4.3) is piecewise affine with derivatives satisfying the above property.

Proof. The result follows directly from Theorems 3.1 and 4.3. \square

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REFERENCES

- [1] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.
- [2] L. Breiman. *Probability*, volume 7 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. Corrected reprint of the 1968 original.
- [3] J. Buzzi. Specification on the interval. *Trans. Amer. Math. Soc.*, 349(7):2737–2754, 1997.
- [4] P. C  nac, B. Chauvin, F. Paccaut, and N. Pouyanne. *Variable length Markov chains and dynamical sources*. S  minaire de Probabilit  s 2010. Springer-Verlag, Berlin, to appear.
- [5] F. Comets, R. Fern  ndez, and P. Ferrari. Processes with long memory: Regenerative construction and perfect simulation. *Ann. of Appl. Prob.*, 12(3):921–943, 2002.
- [6] W. Doeblin and R. Fortet. Sur les cha  nes    liaisons compl  tes. *Bull. Soc. Math. France*, 65:132–148, 1937.

- [7] R. Fernández and G. Maillard. Chains with complete connections and one-dimensional Gibbs measures. *Electron. J. Probab.*, 9:no. 6, 145–176 (electronic), 2004.
- [8] T. E. Harris. On chains of infinite order. *Pacific J. Math.*, 5:707–24, 1955.
- [9] F. Hofbauer and G. Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.*, 180(1):119–140, 1982.
- [10] F. Ledrappier. Principe variationnel et systmes dynamiques symboliques. *Probability Theory and Related Fields*, 30:185–202, 1974. 10.1007/BF00533471.
- [11] C. Liverani. Decay of correlations for piecewise expanding maps. *J. Statist. Phys.*, 78(3-4):1111–1129, 1995.
- [12] O. Onicescu and G. Mihoc. Sur les chaînes statistiques. *C. R. Acad. Sci. Paris*, 200:511–12, 1935a.
- [13] J. Rissanen. A universal data compression system. *IEEE Trans. Inform. Theory*, 29(5):656–664, 1983.
- [14] D. Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.
- [15] J. G. Sinaï. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
- [16] P. Walters. Equilibrium states for β -transformations and related transformations. *Mathematische Zeitschrift*, 159:65–88, 1978.
- [17] P. Walters. Invariant measures and equilibrium states for some mappings which expand distances. *Trans. Amer. Math. Soc.*, 236:121153, 1978.

CENTRE DE PHYSIQUE THÉORIQUE, CNRS UMR 7644, ECOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE

E-mail address: `collet@cpht.polytechnique.fr`

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, BP 66281, 05315-970 SÃO PAULO, BRASIL

E-mail address: `galves@usp.br`